Problem 2.62

Suppose the bottom of the infinite square well is not flat ($V(x) = 0$), but rather

$$V(x) = 500V_0 \sin\left(\frac{\pi x}{a}\right), \quad \text{where} \quad V_0 \equiv \frac{\hbar^2}{2ma^2}.$$

Use the method of Problem 2.61 to find the three lowest allowed energies numerically, and plot the associated wave functions (use $N = 100$).

Solution

Begin with the Schrödinger equation.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t), \quad -\infty < x < \infty,\ t > 0$$

The modified infinite square well potential is

$$V(x,t) = \begin{cases} \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right) & \text{if } 0 \le x \le a \\ \infty & \text{else} \end{cases}.$$

Split up the PDE over the intervals where the potential is defined.

$$\begin{cases} i\hbar\dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + (\infty)\Psi(x,t), \quad x < 0,\ x > a \\ i\hbar\dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2 \Psi}{\partial x^2} + \dfrac{500\hbar^2}{2ma^2}\sin\left(\dfrac{\pi x}{a}\right)\Psi(x,t), \quad 0 \le x \le a \end{cases}$$

The only way to satisfy the PDE on top is to have $\Psi(x,t) = 0$. The fact that the wave function must be continuous leads to the two Dirichlet boundary conditions, $\Psi(0,t) = 0$ and $\Psi(a,t) = 0$, for the remaining PDE.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right)\Psi(x,t), \quad 0 \le x \le a,\ t > 0$$

$$\Psi(0,t) = 0$$

$$\Psi(a,t) = 0$$

Apply the method of separation of variables since the PDE and its associated boundary conditions are linear and homogeneous: Assume a product solution for $\Psi(x,t) = \phi(t)\psi(x)$ and plug it into the PDE

$$i\hbar\frac{\partial}{\partial t}[\phi(t)\psi(x)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\phi(t)\psi(x)] + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right)[\phi(t)\psi(x)]$$

$$\rightarrow \quad i\hbar\psi(x)\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\phi(t)\frac{d^2\psi}{dx^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right)\phi(t)\psi(x)$$

and the boundary conditions.

$$\Psi(0,t) = 0 \quad \rightarrow \quad \phi(t)\psi(0) = 0 \quad \rightarrow \quad \psi(0) = 0$$

$$\Psi(a,t) = 0 \quad \rightarrow \quad \phi(t)\psi(a) = 0 \quad \rightarrow \quad \psi(a) = 0$$

Divide both sides of the PDE by $\phi(t)\psi(x)$ to separate variables.

$$i\hbar\frac{1}{\phi(t)}\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right)$$

The only way a function of t can be equal to a function of x is if both are equal to a constant.

$$i\hbar\frac{1}{\phi(t)}\frac{d\phi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right) = E$$

As a result of separating variables, Schrödinger's equation has reduced to two ODEs, one in t and one in x.

$$\left.\begin{aligned} i\hbar\frac{1}{\phi(t)}\frac{d\phi}{dt} &= E \\ -\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi}{dx^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right) &= E \end{aligned}\right\}$$

The ODE in x is called the time-independent Schrödinger equation (TISE).

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x}{a}\right)\psi(x) = E\psi(x), \quad \psi(0) = 0,\ \psi(a) = 0$$

Apply the method of finite differences in order to solve it: Replace the second derivative with a centered second difference and discretize the $0 \le x \le a$ interval into $N+1$ equal pieces. The continuous variable x becomes a discrete one $x_j = j\Delta x$, where $j = 0, 1, \ldots, N+1$ and $\Delta x = a/(N+1)$ is the mesh size.

$$-\frac{\hbar^2}{2m}\left(\frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2}\right) + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{\pi x_j}{a}\right)\psi(x_j) = E\psi(x_j), \quad \psi(x_0) = 0,\ \psi(x_{N+1}) = 0$$

$$\frac{\hbar^2}{2m(\Delta x)^2}(-\psi_{j+1} + 2\psi_j - \psi_{j-1}) + \frac{500\hbar^2}{2ma^2}\sin\left(\frac{j\pi\Delta x}{a}\right)\psi_j = E\psi_j, \quad \psi_0 = 0,\ \psi_{N+1} = 0$$

The scheme for the unknowns is therefore

$$\boxed{\frac{(N+1)^2\hbar^2}{2ma^2}\left\{-\psi_{j+1} + \left[2 + \frac{500}{(N+1)^2}\sin\left(\frac{j\pi}{N+1}\right)\right]\psi_j - \psi_{j-1}\right\} = E\psi_j, \quad j = 1, 2, \ldots, N.}$$

For $N = 1$,

$$\frac{2\hbar^2}{ma^2}(-\psi_2 + 127\psi_1 - \psi_0) = E\psi_1 \quad \Rightarrow \quad \frac{2\hbar^2}{ma^2}\begin{bmatrix}127\end{bmatrix}\begin{bmatrix}\psi_1\end{bmatrix} = E\begin{bmatrix}\psi_1\end{bmatrix} \quad \Rightarrow \quad \mathsf{H} = \frac{2\hbar^2}{ma^2}\begin{bmatrix}127\end{bmatrix}.$$

For $N = 2$,

$$\left.\begin{aligned} \frac{9\hbar^2}{2ma^2}\left(-\psi_2 + \frac{6\sqrt{3}+250}{3\sqrt{3}}\psi_1 - \psi_0\right) &= E\psi_1 \\ \frac{9\hbar^2}{2ma^2}\left(-\psi_3 + \frac{6\sqrt{3}+250}{3\sqrt{3}}\psi_2 - \psi_1\right) &= E\psi_2 \end{aligned}\right\} \quad \Rightarrow \quad \frac{9\hbar^2}{2ma^2}\begin{bmatrix}\frac{6\sqrt{3}+250}{3\sqrt{3}} & -1 \\ -1 & \frac{6\sqrt{3}+250}{3\sqrt{3}}\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = E\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}$$

$$\Rightarrow \quad \mathsf{H} = \frac{9\hbar^2}{2ma^2}\begin{bmatrix}\frac{6\sqrt{3}+250}{3\sqrt{3}} & -1 \\ -1 & \frac{6\sqrt{3}+250}{3\sqrt{3}}\end{bmatrix}.$$

For $N = 3$,

$$\left.\begin{aligned} \frac{8\hbar^2}{ma^2}\left(-\psi_2 + \frac{8\sqrt{2}+125}{4\sqrt{2}}\psi_1 - \psi_0\right) &= E\psi_1 \\ \frac{8\hbar^2}{ma^2}\left(-\psi_3 + \frac{133}{4}\psi_2 - \psi_1\right) &= E\psi_2 \\ \frac{8\hbar^2}{ma^2}\left(-\psi_4 + \frac{8\sqrt{2}+125}{4\sqrt{2}}\psi_3 - \psi_2\right) &= E\psi_3 \end{aligned}\right\} \quad\Rightarrow\quad \frac{8\hbar^2}{ma^2}\begin{bmatrix} \frac{8\sqrt{2}+125}{4\sqrt{2}} & -1 & 0 \\ -1 & \frac{133}{4} & -1 \\ 0 & -1 & \frac{8\sqrt{2}+125}{4\sqrt{2}} \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = E\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix}$$

$$\Rightarrow\quad \mathsf{H} = \frac{8\hbar^2}{ma^2}\begin{bmatrix} \frac{8\sqrt{2}+125}{4\sqrt{2}} & -1 & 0 \\ -1 & \frac{133}{4} & -1 \\ 0 & -1 & \frac{8\sqrt{2}+125}{4\sqrt{2}} \end{bmatrix}.$$

For $N = 4$,

$$\left.\begin{aligned} \frac{25\hbar^2}{2ma^2}\left[-\psi_2 + \left(2+5\sqrt{10-2\sqrt{5}}\right)\psi_1 - \psi_0\right] &= E\psi_1 \\ \frac{25\hbar^2}{2ma^2}\left[-\psi_3 + \left(2+5\sqrt{10+2\sqrt{5}}\right)\psi_2 - \psi_1\right] &= E\psi_2 \\ \frac{25\hbar^2}{2ma^2}\left[-\psi_4 + \left(2+5\sqrt{10+2\sqrt{5}}\right)\psi_3 - \psi_2\right] &= E\psi_3 \\ \frac{25\hbar^2}{2ma^2}\left[-\psi_5 + \left(2+5\sqrt{10-2\sqrt{5}}\right)\psi_4 - \psi_3\right] &= E\psi_4 \end{aligned}\right\}$$

$$\Rightarrow\quad \frac{25\hbar^2}{2ma^2}\begin{bmatrix} 2+5\sqrt{10-2\sqrt{5}} & -1 & 0 & 0 \\ -1 & 2+5\sqrt{10+2\sqrt{5}} & -1 & 0 \\ 0 & -1 & 2+5\sqrt{10+2\sqrt{5}} & -1 \\ 0 & 0 & -1 & 2+5\sqrt{10-2\sqrt{5}} \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} = E\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}$$

$$\Rightarrow\quad \mathsf{H} = \frac{25\hbar^2}{2ma^2}\begin{bmatrix} 2+5\sqrt{10-2\sqrt{5}} & -1 & 0 & 0 \\ -1 & 2+5\sqrt{10+2\sqrt{5}} & -1 & 0 \\ 0 & -1 & 2+5\sqrt{10+2\sqrt{5}} & -1 \\ 0 & 0 & -1 & 2+5\sqrt{10-2\sqrt{5}} \end{bmatrix}.$$

Using the scheme for entries along the main diagonal, the H matrix can be constructed for any value of N. For $N = 11$,

$$\mathsf{H} = \frac{72\hbar^2}{ma^2}\begin{bmatrix} \frac{144+125\sqrt{2-\sqrt{3}}}{72} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{269}{72} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{72\sqrt{2}+125}{36\sqrt{2}} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \frac{48\sqrt{3}+125}{24\sqrt{3}} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{144+125\sqrt{2+\sqrt{3}}}{72} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \frac{197}{36} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \frac{144+125\sqrt{2+\sqrt{3}}}{72} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{48\sqrt{3}+125}{24\sqrt{3}} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{72\sqrt{2}+125}{36\sqrt{2}} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{269}{72} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{144+125\sqrt{2-\sqrt{3}}}{72} \end{bmatrix}.$$

In any case, the scheme is an (approximate) eigenvalue problem that becomes more accurate the higher N is.

$$\mathsf{H}\Psi = E\Psi$$

The plan is to plug all these H matrices into Mathematica to obtain their lowest three eigenvalues and corresponding eigenvectors. Recall the following numerical integration formulas.

Trapezoidal Rule: $$\int_0^a f(x)\,dx \approx \frac{a}{2(N+1)}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_N) + f(x_{N+1})\right]$$

Simpson's Rule: $$\int_0^a f(x)\,dx \approx \frac{a}{3(N+1)}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{N-1}) + 4f(x_N) + f(x_{N+1})\right]$$

The latter is usually more accurate but can only be used if $N + 1$ is even.

Use Mathematica to obtain the lowest three eigenvalues for many values of N. Set $\hbar = 1$, $m = 1$, and $a = 1$ and enclose any square roots within `N[ ]` in order to decrease the computation time.

$$N = 1: \qquad E = \left\{\frac{254\hbar^2}{ma^2}\right\}$$

$$N = 2: \qquad E = \left\{\frac{221.0063509461097\hbar^2}{ma^2}, \frac{230.00635094610968\hbar^2}{ma^2}\right\}$$

$$N = 3: \qquad E = \left\{\frac{191.06846915268133\hbar^2}{ma^2}, \frac{192.77669529663686\hbar^2}{ma^2}, \frac{267.70822614395547\hbar^2}{ma^2}\right\}$$

$$N = 4: \qquad E = \left\{\frac{169.99962442556728\hbar^2}{ma^2}, \frac{170.45550069989838\hbar^2}{ma^2}, \frac{252.21081772133937\hbar^2}{ma^2}\right\}$$

$$N = 5: \qquad E = \left\{\frac{157.4025893324986\hbar^2}{ma^2}, \frac{157.5865906224864\hbar^2}{ma^2}, \frac{241.84025673580197\hbar^2}{ma^2}\right\}$$

$$N = 7: \qquad E = \left\{\frac{147.6936481649067\hbar^2}{ma^2}, \frac{147.770719079939\hbar^2}{ma^2}, \frac{230.05384756814834\hbar^2}{ma^2}\right\}$$

$$N = 9: \qquad E = \left\{\frac{146.86587765219653\hbar^2}{ma^2}, \frac{146.92899178222265\hbar^2}{ma^2}, \frac{228.4516342740409\hbar^2}{ma^2}\right\}$$

$$N = 11: \qquad E = \left\{\frac{147.9357610474816\hbar^2}{ma^2}, \frac{147.99854447401353\hbar^2}{ma^2}, \frac{230.74490111020734\hbar^2}{ma^2}\right\}$$

$$N = 13: \qquad E = \left\{\frac{148.97317718115622\hbar^2}{ma^2}, \frac{149.03744807044146\hbar^2}{ma^2}, \frac{232.8359310028265\hbar^2}{ma^2}\right\}$$

$$N = 16: \qquad E = \left\{\frac{150.0327569294224\hbar^2}{ma^2}, \frac{150.0989089271708\hbar^2}{ma^2}, \frac{234.71531702079255\hbar^2}{ma^2}\right\}$$

$$N = 21: \qquad E = \left\{\frac{150.97804115333088\hbar^2}{ma^2}, \frac{151.04595438006479\hbar^2}{ma^2}, \frac{236.19805241042846\hbar^2}{ma^2}\right\}$$

$$N = 31: \qquad E = \left\{\frac{151.7264624934166\hbar^2}{ma^2}, \frac{151.79576654646766\hbar^2}{ma^2}, \frac{237.27003146074932\hbar^2}{ma^2}\right\}$$

$$N = 51: \qquad E = \left\{\frac{152.14336176546368\hbar^2}{ma^2}, \frac{152.21342869775268\hbar^2}{ma^2}, \frac{237.83518133953876\hbar^2}{ma^2}\right\}$$

$$N = 101: \qquad E = \left\{\frac{152.33109043657436\hbar^2}{ma^2}, \frac{152.40149690859414\hbar^2}{ma^2}, \frac{238.0830672534101\hbar^2}{ma^2}\right\}$$

Now the eigenvectors will be found using Mathematica. Like before, set $\hbar = 1$, $m = 1$, and $a = 1$ and enclose any square roots within N[ ] in order to decrease the computation time. Plugging the H matrix for $N = 5$ into Mathematica yields the following eigenvectors.

$$\Psi_1 = \begin{bmatrix} 0.692873 \\ 0.138475 \\ 0.0387652 \\ 0.138475 \\ 0.692873 \end{bmatrix} \qquad \Psi_2 = \begin{bmatrix} -0.694726 \\ -0.131743 \\ 0 \\ 0.131743 \\ 0.694726 \end{bmatrix} \qquad \Psi_3 = \begin{bmatrix} -0.133936 \\ 0.601521 \\ 0.490374 \\ 0.601521 \\ -0.133936 \end{bmatrix}$$

These eigenvectors are unique up to a multiplicative constant.

$$\Psi_1 = \begin{bmatrix} 0.692873A_1 \\ 0.138475A_1 \\ 0.0387652A_1 \\ 0.138475A_1 \\ 0.692873A_1 \end{bmatrix} \qquad \Psi_2 = \begin{bmatrix} -0.694726A_2 \\ -0.131743A_2 \\ 0 \\ 0.131743A_2 \\ 0.694726A_2 \end{bmatrix} \qquad \Psi_3 = \begin{bmatrix} -0.133936A_3 \\ 0.601521A_3 \\ 0.490374A_3 \\ 0.601521A_3 \\ -0.133936A_3 \end{bmatrix}$$

These constants are chosen so that the eigenvectors are normalized. Apply Simpson's rule:

$$1 = \int_0^a |\psi|^2\,dx \approx \frac{a}{3(5+1)}\left\{\underbrace{[\psi(x_0)]^2}_{=\,0} + 4[\psi(x_1)]^2 + 2[\psi(x_2)]^2 + 4[\psi(x_3)]^2 + 2[\psi(x_4)]^2 + 4[\psi(x_5)]^2 + \underbrace{[\psi(x_6)]^2}_{=\,0}\right\}.$$

$$\begin{cases} 1 = \dfrac{a}{18}\left[4(0.692873A_1)^2 + 2(0.138475A_1)^2 + 4(0.0387652A_1)^2 + 2(0.138475A_1)^2 + 4(0.692873A_1)^2\right] \\ 1 = \dfrac{a}{18}\left[4(-0.694726A_2)^2 + 2(-0.131743A_2)^2 + 4(0)^2 + 2(0.131743A_2)^2 + 4(0.694726A_2)^2\right] \\ 1 = \dfrac{a}{18}\left[4(-0.133936A_3)^2 + 2(0.601521A_3)^2 + 4(0.490374A_3)^2 + 2(0.601521A_3)^2 + 4(-0.133936A_3)^2\right] \end{cases}$$

Solve these equations for A_1, A_2, and A_3.

$$A_1 = \pm\frac{2.14196}{\sqrt{a}} \qquad A_2 = \pm\frac{2.13997}{\sqrt{a}} \qquad A_3 = \pm\frac{2.65545}{\sqrt{a}}$$

As a result, the normalized eigenvectors for $N = 5$ are

$$\Psi_1 = \frac{1}{\sqrt{a}}\begin{bmatrix} 1.4841 \\ 0.296607 \\ 0.0830333 \\ 0.296607 \\ 1.4841 \end{bmatrix} \qquad \Psi_2 = \frac{1}{\sqrt{a}}\begin{bmatrix} 1.48669 \\ 0.281927 \\ 0 \\ -0.281927 \\ -1.48669 \end{bmatrix} \qquad \Psi_3 = \frac{1}{\sqrt{a}}\begin{bmatrix} 0.355659 \\ -1.59731 \\ -1.30216 \\ -1.59731 \\ 0.355659 \end{bmatrix}.$$

And these each lead to seven points that approximate the first three eigenstates.

$$\psi_1(x): \quad \left\{(0,0), \left(\frac{a}{6}, \frac{1.4841}{\sqrt{a}}\right), \left(\frac{a}{3}, \frac{0.296607}{\sqrt{a}}\right), \left(\frac{a}{2}, \frac{0.0830333}{\sqrt{a}}\right), \left(\frac{2a}{3}, \frac{0.296607}{\sqrt{a}}\right), \left(\frac{5a}{6}, \frac{1.4841}{\sqrt{a}}\right), (a,0)\right\}$$

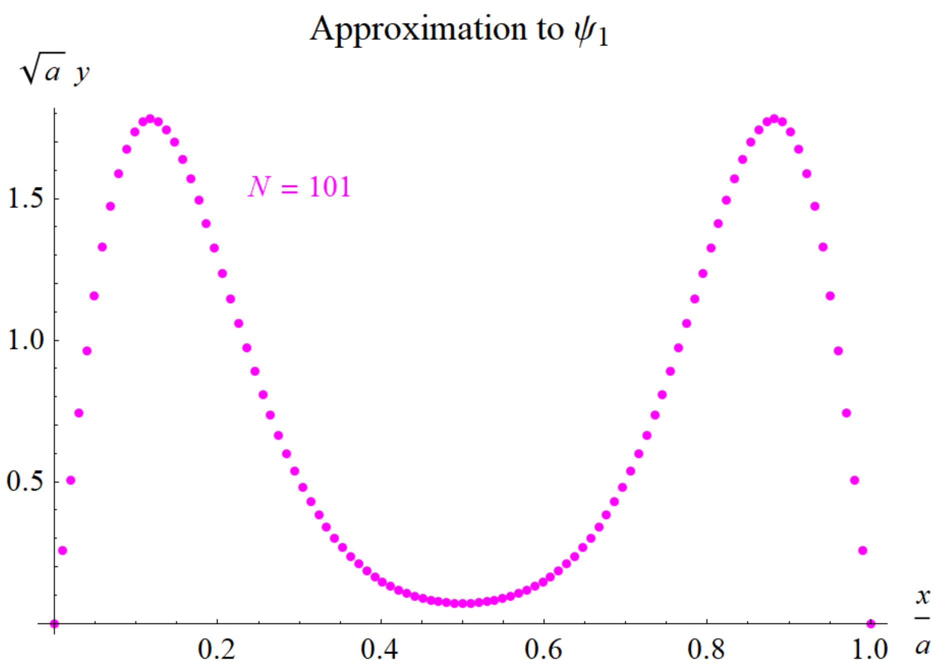
$$\psi_2(x): \quad \left\{(0,0), \left(\frac{a}{6}, \frac{1.48669}{\sqrt{a}}\right), \left(\frac{a}{3}, \frac{0.281927}{\sqrt{a}}\right), \left(\frac{a}{2}, 0\right), \left(\frac{2a}{3}, -\frac{0.281927}{\sqrt{a}}\right), \left(\frac{5a}{6}, -\frac{1.48669}{\sqrt{a}}\right), (a,0)\right\}$$

$$\psi_3(x): \quad \left\{(0,0), \left(\frac{a}{6}, \frac{0.355659}{\sqrt{a}}\right), \left(\frac{a}{3}, -\frac{1.59731}{\sqrt{a}}\right), \left(\frac{a}{2}, -\frac{1.30216}{\sqrt{a}}\right), \left(\frac{2a}{3}, -\frac{1.59731}{\sqrt{a}}\right), \left(\frac{5a}{6}, \frac{0.355659}{\sqrt{a}}\right), (a,0)\right\}$$

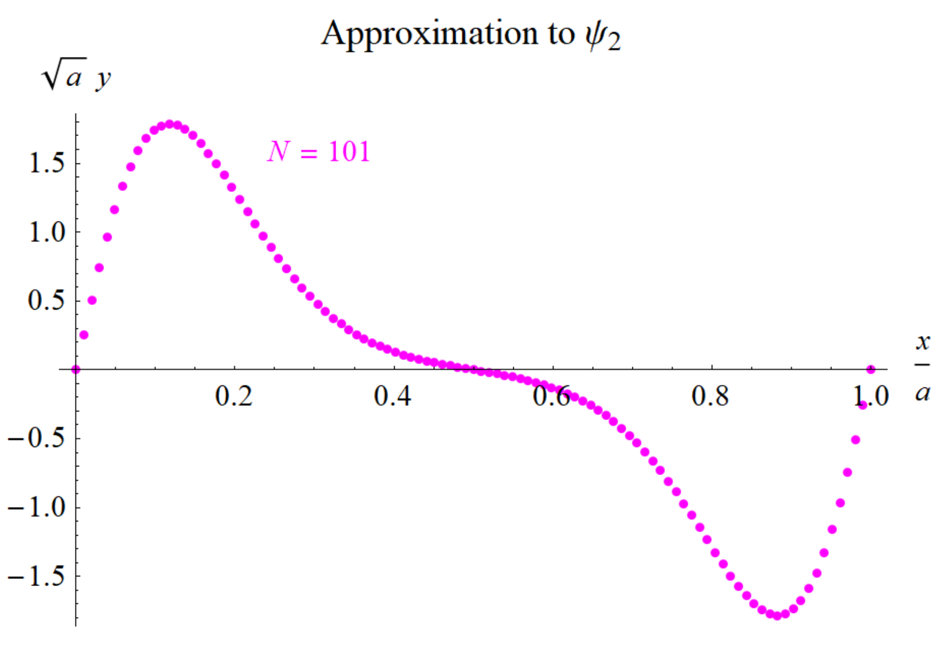
Use this same procedure to find the normalized eigenvectors for $N = 101$.

$$\Psi_1 = \frac{1}{\sqrt{a}}\begin{bmatrix} 0.257309 \\ 0.507465 \\ 0.744261 \\ 0.962564 \\ 1.15836 \\ 1.32878 \\ 1.47202 \\ 1.5873 \\ 1.67469 \\ 1.73507 \\ 1.76992 \\ 1.78122 \\ 1.77127 \\ 1.74264 \\ 1.69798 \\ 1.63997 \\ 1.57123 \\ 1.49423 \\ 1.41127 \\ 1.32446 \\ 1.23563 \\ 1.14641 \\ 1.05816 \\ 0.972009 \\ 0.888864 \\ 0.809428 \\ 0.734215 \\ 0.663579 \\ 0.597729 \\ 0.536755 \\ 0.480648 \\ 0.42932 \\ 0.38262 \\ 0.340349 \\ 0.302277 \\ 0.268152 \\ 0.23771 \\ 0.210685 \\ 0.186816 \\ 0.165848 \\ 0.14754 \\ 0.131669 \\ 0.118028 \\ 0.106433 \\ 0.0967166 \\ 0.0887373 \\ 0.0823737 \\ 0.0775266 \\ 0.0741193 \\ 0.0720968 \\ 0.0714262 \\ 0.0720968 \\ 0.0741193 \\ 0.0775266 \\ 0.0823737 \\ 0.0887373 \\ 0.0967166 \\ 0.106433 \\ 0.118028 \\ 0.131669 \\ 0.14754 \\ 0.165848 \\ 0.186816 \\ 0.210685 \\ 0.23771 \\ 0.268152 \\ 0.302277 \\ 0.340349 \\ 0.38262 \\ 0.42932 \\ 0.480648 \\ 0.536755 \\ 0.597729 \\ 0.663579 \\ 0.734215 \\ 0.809428 \\ 0.888864 \\ 0.972009 \\ 1.05816 \\ 1.14641 \\ 1.23563 \\ 1.32446 \\ 1.41127 \\ 1.49423 \\ 1.57123 \\ 1.63997 \\ 1.69798 \\ 1.74264 \\ 1.77127 \\ 1.78122 \\ 1.76992 \\ 1.73507 \\ 1.67469 \\ 1.5873 \\ 1.47202 \\ 1.32878 \\ 1.15836 \\ 0.962564 \\ 0.744261 \\ 0.507465 \\ 0.257309 \end{bmatrix} \quad \Psi_2 = \frac{1}{\sqrt{a}}\begin{bmatrix} 0.257895 \\ 0.508616 \\ 0.745941 \\ 0.96472 \\ 1.16093 \\ 1.33169 \\ 1.4752 \\ 1.59065 \\ 1.67815 \\ 1.73856 \\ 1.77336 \\ 1.78453 \\ 1.7744 \\ 1.74553 \\ 1.70058 \\ 1.64224 \\ 1.57312 \\ 1.49572 \\ 1.41233 \\ 1.32507 \\ 1.23577 \\ 1.14607 \\ 1.05731 \\ 0.970641 \\ 0.886961 \\ 0.806968 \\ 0.731176 \\ 0.659932 \\ 0.593438 \\ 0.531777 \\ 0.474931 \\ 0.422801 \\ 0.375223 \\ 0.331983 \\ 0.292834 \\ 0.257505 \\ 0.225709 \\ 0.197156 \\ 0.171551 \\ 0.148609 \\ 0.128049 \\ 0.109602 \\ 0.0930101 \\ 0.0780281 \\ 0.0644233 \\ 0.0519745 \\ 0.0404712 \\ 0.0297126 \\ 0.0195052 \\ 0.00966209 \\ 0 \\ -0.00966209 \\ -0.0195052 \\ -0.0297126 \\ -0.0404712 \\ -0.0519745 \\ -0.0644233 \\ -0.0780281 \\ -0.0930101 \\ -0.109602 \\ -0.128049 \\ -0.148609 \\ -0.171551 \\ -0.197156 \\ -0.225709 \\ -0.257505 \\ -0.292834 \\ -0.331983 \\ -0.375223 \\ -0.422801 \\ -0.474931 \\ -0.531777 \\ -0.593438 \\ -0.659932 \\ -0.731176 \\ -0.806968 \\ -0.886961 \\ -0.970641 \\ -1.05731 \\ -1.14607 \\ -1.23577 \\ -1.32507 \\ -1.41233 \\ -1.49572 \\ -1.57312 \\ -1.64224 \\ -1.70058 \\ -1.74553 \\ -1.7744 \\ -1.78453 \\ -1.77336 \\ -1.73856 \\ -1.67815 \\ -1.59065 \\ -1.4752 \\ -1.33169 \\ -1.16093 \\ -0.96472 \\ -0.745941 \\ -0.508616 \\ -0.257895 \end{bmatrix} \quad \Psi_3 = \frac{1}{\sqrt{a}}\begin{bmatrix} 0.184114 \\ 0.360074 \\ 0.52062 \\ 0.659647 \\ 0.772379 \\ 0.855455 \\ 0.906933 \\ 0.926227 \\ 0.913988 \\ 0.871938 \\ 0.802685 \\ 0.709515 \\ 0.596191 \\ 0.466749 \\ 0.32532 \\ 0.17597 \\ 0.0225682 \\ -0.131325 \\ -0.282529 \\ -0.428303 \\ -0.566368 \\ -0.694914 \\ -0.812592 \\ -0.918488 \\ -1.01208 \\ -1.09322 \\ -1.16204 \\ -1.21895 \\ -1.26455 \\ -1.29963 \\ -1.32506 \\ -1.34183 \\ -1.35094 \\ -1.35342 \\ -1.35029 \\ -1.34253 \\ -1.33108 \\ -1.31683 \\ -1.30058 \\ -1.2831 \\ -1.26505 \\ -1.24704 \\ -1.22959 \\ -1.21318 \\ -1.19819 \\ -1.18497 \\ -1.1738 \\ -1.16488 \\ -1.15839 \\ -1.15445 \\ -1.15313 \\ -1.15445 \\ -1.15839 \\ -1.16488 \\ -1.1738 \\ -1.18497 \\ -1.19819 \\ -1.21318 \\ -1.22959 \\ -1.24704 \\ -1.26505 \\ -1.2831 \\ -1.30058 \\ -1.31683 \\ -1.33108 \\ -1.34253 \\ -1.35029 \\ -1.35342 \\ -1.35094 \\ -1.34183 \\ -1.32506 \\ -1.29963 \\ -1.26455 \\ -1.21895 \\ -1.16204 \\ -1.09322 \\ -1.01208 \\ -0.918488 \\ -0.812592 \\ -0.694914 \\ -0.566368 \\ -0.428303 \\ -0.282529 \\ -0.131325 \\ 0.0225682 \\ 0.17597 \\ 0.32532 \\ 0.466749 \\ 0.596191 \\ 0.709515 \\ 0.802685 \\ 0.871938 \\ 0.913988 \\ 0.926227 \\ 0.906933 \\ 0.855455 \\ 0.772379 \\ 0.659647 \\ 0.52062 \\ 0.360074 \\ 0.184114 \end{bmatrix}$$

Below is a plot of the points implied by Ψ_1 for $N = 101$; these points serve as an approximation to the first eigenstate.



Below is a plot of the points implied by Ψ_2 for $N = 101$; these points serve as an approximation to the second eigenstate.



Below is a plot of the points implied by Ψ_3 for $N = 101$; these points serve as an approximation to the third eigenstate.